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ON FOURIER TRANSFORMS AND ITS EXTENSION TO A SPACE OF GENERALIZED FUNCTIONS

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ABSTRACT : Pseudoquotients generalize the field of quotients of integral domains. Many integral transforms are widely extended to Boehmians but a few to pseudoqoutients. In this paper we display an idea of research. We consider certain space of pseudoquotients. The Fourier transform of a pseudoquotient in the proposed space is introduced as a pseudoquotient in the same space. Some properties are established.

Keywords : Pseudoquotients; Fourier Transform; Distribution; Boehmian.

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1. INTRODUCTION

The set of natural numbers $\mathbb{Q}$ can be thought as the minimal extention of the ring of integers $\mathbb{Z}$. Two pairs (p,q) and (r,s) where $p,r \in \mathbb{Z}$, and $q,s \in \mathbb{N}$, the set of natural numbers, are said to be equivalent, denoted by $(p,q)\sim(r,s)$ (r,s), if $sp = rq$. Then $\mathbb{Q}$ is given as the set of equivalence classes of pairs $[(p,q)]$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ which leads to an integral domain.

The concept of integral domains lead Boehme, T.K. [14] to the idea of regular operators which has been motivated to the concept of Boehmian spaces [6]. The space of Boehmians is constructed using an algebraic approach, which utilizes convolution and approximate identities or delta sequences. A proper subspace can be identified with the space of distributions. In [8,9,10,11,12,13,15,16,17,18,19] integral transforms found their application to various spaces of Boehmian. In the sequence of these integral transforms, Mikusinski, P. [12] first extended the Fourier transform

$$Kf\;(\xi)=\int_{-\infty}^{\infty} f\;(x)e^{-ix\xi}dx$$

to a space β_ℓ of integrable Boehmians by the limit

$$K\left[\frac{f_n}{\delta_n}\right]=\lim_{n\to\infty} Kf_n$$

where convergence is considered over compact subsets of $\mathbb{R}$. Later, Fourier transforms have then been given various forms by Karunakarana, V. and Ganesana, C. in [11] and Nemzer in [13].

A special class of the abstract Boehmians is the class of pseudoquotients. Pseudoquotients are simpler than general Boehmians and have desirable properties.

The general construction is as follows.

Theorem 1 Let G be a commutative semigroup acting on X injectively. Then the operation

$$(x,\varphi)\sim(y,\psi) \text{ iff } \psi x=\varphi y \tag{1}$$

where $(x,\varphi),(y,\psi)\in X\times G$ generalizes to an equivalence relation. The space of all equivalence classes is denoted by $B(\mathrm{X},G,\sim)$ whose elements are called generalized quotients or pseudoquotients; see [1] and [2]. Elements of X are identified with elements of $B(\mathrm{X},G,\sim)$ via the embedding

$$i:X\to B(\mathrm{X},G,\sim)$$

$$i(x)=\frac{\varphi x}{\varphi} \tag{2}$$

The action of G is extended to $B(\mathrm{X},G,\sim)$ by the formula

$$\varphi\frac{x}{\psi}=\frac{\varphi x}{\psi} \tag{3}$$

Then it is seen that

$$\varphi\frac{x}{\psi}=\frac{\varphi x}{\psi}=x \tag{4}$$

Let $(\mathrm{X},*)$ be commutative group and G be a commutative semigroup of injective homomorphisms on X, then $B(\mathrm{X},G,\sim)$ is a commutative group with the group operations

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$$\frac{x}{\varphi}*\frac{y}{\psi}=\frac{\psi x*\varphi y}{\varphi*\psi} \qquad (5)$$

Theorem 2 . If X is a vector space and G is a commutative semigroup of injective linear mappings from X into X , then $B(\mathrm{X},G,\sim)$ is a vector space with the operations

$$\frac{x}{\varphi}+\frac{y}{\psi}=\frac{\psi x+\varphi y}{\varphi\psi} \qquad (6)$$

And

$$\lambda\frac{x}{\varphi}=\frac{\lambda x}{\varphi}.$$

2. PSEUDOQOUTIENTS OF RAPID DESCENTS

Let S denote the linear space of rapid descents whose elements are smooth functions and decay to zero faster than every power of t . When t is one dimensional, every function $\varphi(t)$ in S satisfies the infinite set of inequalities

$$\left|t^{m}\varphi^{(k)}(t)\right|\leq b_{m,k}, t\in\mathbb{R},$$

where m and k run through nonegative integers. The above expression can be interpreted to mean

$$\lim t^{m}\varphi^{(k)}(t)=0.$$

Members of S are testing functions of rapid descent or rapidly decreasing functions. The dual space S' of S is the space of distributions of slow growth . We refer reader for [4] for more properties of S and S'.

Let $S=\mathbb{R}$ and $G=S$ and $\sim$ is defined by

$$\frac{x}{\varphi}\sim\frac{y}{\psi} \text{ if } \psi x=\varphi y \qquad (7)$$

where $y,x\in\mathbb{R}$ and $\varphi,\psi\in S$, then the pseudoquotient space $B(\mathbb{R},S,\sim)$ is ofcourse of rapid descents.

Each $x\in\mathbb{R}$ is identified in $B(\mathbb{R},S,\sim)$ via the embedding

$$i:\mathbb{R}\to B(\mathbb{R},S,\sim)$$

$$i(x)=\frac{\varphi x}{\varphi}. \qquad (8)$$

The action of S may be extended to $B(\mathbb{R},S,\sim)$ by $\varphi\frac{x}{\psi}=\frac{\varphi x}{\psi},\varphi,\psi\in S,\mathrm{x}\in\mathbb{R}.$ Then it gives

$$\varphi\frac{x}{\varphi}=\frac{\varphi x}{\varphi}=x\ .$$

Theorem 3 Let $\varphi\in S$ then $K\varphi\in S$.

Proof of this theorem can be obtained from [5].

Details are thus avoided.

The pesudoqoutient space $B(\mathbb{R},S,\sim)$ is a commutative group with the group operations

$$\frac{x}{\varphi}*\frac{y}{\psi}=\frac{\psi x*\varphi y}{\varphi\psi}.$$

where $*$ is the usual product on $\mathbb{R}$. Further, $B\left(\mathbb{R},S,\sim\right)$ is a vector space with the operations

$$\frac{x}{\varphi}+\frac{y}{\psi}=\frac{\psi x*\varphi y}{\varphi\psi} \text{ and } \lambda\frac{x}{\varphi}=\frac{\lambda x}{\varphi}.$$

3. FOURIER TRANSFORMS OF PSEUDOQOUTIENTS

Definition 4 Let $\frac{x}{\varphi}\in B\left(\mathbb{R},S,\sim\right)$ then we introduce the Fourier transform of $\frac{x}{\varphi}$ (x/ϕ) as the extension of K defined by

$$K\frac{x}{\varphi}=\frac{Kx}{\varphi}. \tag{9}$$

where $x\in\mathbb{R}$ and $\varphi\in S$.

Righthand side of (9), belongs to $B\left(\mathbb{R},S,\sim\right)$ by Theorem 3 .

Theorem 5 The extended Fourier transform K is well defined mapping from $B\left(\mathbb{R},S,\sim\right)$ into $B\left(\mathbb{R},S,\sim\right)$.

Proof Let $\frac{x}{\varphi}=\frac{y}{\psi}$ in the sense of $B\left(\mathbb{R},S,\sim\right)$ then (7) implies

$$\psi x=\varphi y\quad. \tag{10}$$

Applying the Fourier transform to (10) yields

$$K\psi x=K\varphi y. \tag{11}$$

The mappings $K\psi,K\varphi\in S$, by Theorem 3. Hence, by (7), (11) gives

$$\frac{x}{K\varphi}=\frac{y}{K\psi}. \tag{12}$$

Thus using (9),(12) then gives

$$K\frac{x}{\varphi}=K\frac{y}{\psi}.$$

Hence the theorem is established.

Theorem 6 The extended Fourier transform K is one-to-one mapping from $B\left(\mathbb{R},S,\sim\right)$ into $B\left(\mathbb{R},S,\sim\right)$.

Proof Let $x,y\in\mathbb{R},\varphi,\psi\in S$ and $K\frac{x}{\varphi}=K\frac{y}{\psi}$ then, by (9), we get

$$\frac{x}{K\varphi}=\frac{y}{K\psi}.$$

Using (7) we get $K\psi x=K\varphi y$.

The fact that K is injective implies $\psi x=\varphi y$. Therefore

$$\frac{x}{\varphi}=\frac{y}{\psi}.$$

This proves the theorem.

Theorem 7 The extended transform K is surjective.

Proof of this theorem is straightforword.

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